Equilibrium temperature of a convex body in a free molecular shearing flow

Lars H. Söderholm*
Department of Mechanics, KTH, SE-100 44 Stockholm, Sweden
(Received 18 June 2002; published 24 September 2002)

It is shown that the equilibrium temperature T_w of a high conductivity axially symmetric convex body in a simply shearing gas of temperature T is given by $T_w/T = 1 + (\beta a/4)(p_{xy}/p)\sin^2\theta \sin(2\varphi)$, θ, φ are polar angles of the axis of the body (z is the polar axis). a is a geometric shape factor of the body (which vanishes for a sphere) and β takes the value 1 if only the lowest order Sonine term is retained. p is the pressure and p_{xy} the viscous pressure. The body is assumed small compared to the mean free path, which is small compared to the length scale of the velocity field.

DOI: 10.1103/PhysRevE.66.031204 PACS number(s): 47.45.Dt

I. INTRODUCTION

A small body in a gas with a temperature gradient experiences a force. This is the well-known phenomenon of thermophoresis. The perhaps simplest case is where the body is small compared to the mean free path which in turn is small compared to the length scale of the temperature field. The force is then caused by the non-Maxwellian nature of the molecular distribution function, see Waldmann [1].

In this paper we point to an analogous phenomenon in a shearing gas. For simplicity, we consider the case where the body is small compared to the mean free path which in turn is small compared to the length scale of the velocity field. An exterior transfer of heat to or from the body is shown to be needed to keep its temperature the same as that of the surrounding gas. If there is no such exterior heat transfer, the body will, in a stationary situation, take on a temperature different from that of the surrounding gas. For a conference report of this work, see Ref. [2].

Bell and Schaaf [3]—see also Schaaf [4]—calculated the heat transfer to an infinite circular cylinder with a diameter small compared to the mean free path in a gas with temperature and velocity varying on a scale large compared to the mean free path. They found a contribution to the heat transfer from the shearing of the gas. They also showed that the equilibrium temperature of the body was affected by the shearing of the gas. To the knowledge of the present author, there has been no systematic study of this phenomenon.

Due to the tensorial nature of the shearing, there will be no net heat exchange, when the body is a sphere. For that reason, we consider bodies of arbitrary shape, in particular axially symmetric bodies, in the present work. As the body is small compared to the mean free path it is a good approximation to assume free molecular flow in a region surrounding the body, see Bird [5] and Waldmann [1]. For a general review of free molecular flow, see Schaaf [4], Cercignani [6], and Sone [7]. The body is taken to be convex, so that there are no multiple collisions with its walls.

We use the Maxwell boundary conditions at the surface, assuming that a fraction α_τ of the incoming particles are diffusely reflected, whereas the fraction $(1-\alpha_\tau)$ are specularly reflected. We further assume that the surface of the body is characterized by the energy accommodation coefficient α_e. See Kogan [8] and Eq. (10) below. The gas is taken to be monatomic.

*Email address: lhs@mech.kth.se

II. METHOD OF APPROXIMATION

Let us formally introduce a sphere surrounding the body. The sphere is large compared with the size of the body, but small compared to the mean free path. We can thus neglect mutual collisions of the gas molecules in the sphere (free molecular flow). By the Liouville equation it follows that the distribution function for a molecule colliding with the body has the same value as it had when the molecule entered the sphere.

The molecules, which have collided with the body, give on the surface of the sphere a small, strongly peaked, contribution to the distribution function. After collisions with other molecules they will have a smeared out and very small influence on the ingoing molecules. So for the incoming molecules it is a good approximation to take the distribution function to be unaffected by the presence of the body.

When the body is absent, the problem is well known. As we assume the length scale of the velocity field to be large compared with the mean free path, first-order Chapman-Enskog theory applies.

The distribution function of the outgoing molecules is found from the appropriate boundary conditions. We assume a simple Maxwell type of boundary condition, where a fraction α_τ of the molecules are reflected diffusely as a Maxwellian with temperature T_r and the rest are reflected specularly. Explicitly, the distribution function f on the boundary satisfies ($\mathbf{n}$ is the outward normal of the surface, $\mathbf{C}\cdot\mathbf{n}$ is here taken non-negative)

$$f(\mathbf{C}) = (1-\alpha_\tau) f(\mathbf{C} - 2(\mathbf{C}\cdot\mathbf{n})\mathbf{n}) + \alpha_\tau n_r \left(\frac{m}{2\pi k_B T_r}\right)^{3/2} \times \exp\left(-\frac{mC^2}{2k_B T_r}\right), \tag{1}$$

m is the mass of the molecules, T_r is the temperature of the diffusely reflected molecules, and k_B is the Boltzmann's constant. The parameter n_r is given by conservation of particles, see Eq. (5) below.

III. EQUILIBRIUM TEMPERATURE

A. Net influx of kinetic energy

For a general distribution f, the total flux of incoming particles is (see Kogan [8])

$$N_{(i)} = -\int_{\mathbf{n}\cdot\mathbf{C}<0} (\mathbf{C}\cdot\mathbf{n}) f d^3C. \tag{2}$$

It is now our object to express the net influx of kinetic energy in terms of the distribution function for incoming particles, which, in the approximation considered, is unaffected by the presence of the body.

The Maxwellian distribution is written as

$$f^{(0)} = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\mathcal{C}^2}. \tag{3}$$

Here, n is the number density of molecules, k_B is the Boltzmann constant, T is the temperature, and m is the molecular mass. $\mathcal{C}_i = \sqrt{m/2k_BT}C_i$ is the dimensionless molecular velocity, where C_i is the ordinary molecular velocity. For a Maxwellian distribution $N_{(i)}$ takes the value

$$N_M = n\sqrt{\frac{k_BT}{2\pi m}}. \tag{4}$$

The surface of the body has the temperature T_w. The diffusely reflected molecules have the temperature T_r. Two densities n_w, n_r are defined by

$$N_{(i)} = n_r\sqrt{\frac{k_BT_r}{2\pi m}} = n_w\sqrt{\frac{k_BT_w}{2\pi m}}. \tag{5}$$

The first of these relations ensures conservation of number of particles.

The total influx of kinetic energy is

$$E_{(i)} = -\frac{m}{2}\int_{\mathbf{n}\cdot\mathbf{C}<0} C^2 (\mathbf{C}\cdot\mathbf{n}) f d^3C. \tag{6}$$

In particular, for a Maxwellian it takes the value

$$E_M = 2k_BTN_M. \tag{7}$$

According to the Maxwell boundary conditions (1), the outflux of kinetic energy is given by

$$E_{(r)} = (1-\alpha_\tau)E_{(i)} + \alpha_\tau E_M(T_r). \tag{8}$$

The temperature of the diffusely reflected particles is related to the temperature of the wall via the energy accommodation coefficient α_e. To define it, let us first, in analogy with Eq. (8) introduce

$$E_{(w)} = (1-\alpha_\tau)E_{(i)} + \alpha_\tau E_M(T_w), \tag{9}$$

$$\alpha_e = \frac{E_{(i)} - E_{(r)}}{E_{(i)} - E_{(w)}}. \tag{10}$$

Equation (10) relates T_r and T_w. They coincide when $\alpha_e = 1$.

As a consequence, we obtain the net influx of kinetic energy [we have here used Eq. (5)]

$$E = \alpha_e\alpha_\tau[E_{(i)} - E_M(T_w)] = \alpha_e\alpha_\tau(E_{(i)} - 2k_BT_wN_{(i)}). \tag{11}$$

We have thus been able, for an arbitrary distribution function f, to express the net influx of kinetic energy in terms of the distribution function for incoming particles only.

B. High conductivity body

Let us now assume that α_e and α_τ are constant, and the thermal conductivity of the body is so high that T_w is uniform in the body. The net total influx of kinetic energy to the body is (S is the area of the body)

$$\int E\, ds = \alpha_e\alpha_\tau S[\overline{E_{(i)}} - 2k_BT_w\overline{N_{(i)}}]. \tag{12}$$

The overbar denotes surface average $\overline{f} = S^{-1}\int f\, ds$.

At stationary conditions, the temperature of the body is thus

$$T_w = \frac{\overline{E_{(i)}}}{2k_B\overline{N_{(i)}}}. \tag{13}$$

For an arbitrary distribution function f we have found the equilibrium temperature of the body in terms of the distribution function for incoming particles only, which in the approximation considered is known.

IV. CHAPMAN-ENSKOG DISTRIBUTION FUNCTION

In a pure shearing flow, to first order in the mean free path the distribution function is given by [see Chapman and Cowling [9]; $\hat{B}(\mathcal{C}^2)$ depends on the molecular forces]

$$f = f^{(0)}\left(1 + \hat{B}(\mathcal{C}^2)\mathcal{C}_{\langle i}\mathcal{C}_{j\rangle}\frac{p_{\langle ij\rangle}}{p}\right), \tag{14}$$

$$p_{ij} = -2\mu v_{\langle i,j\rangle}$$

is the viscous pressure tensor, p is the pressure. $\langle\cdots\rangle$ stands for the symmetric traceless part and comma for partial derivative. $\mathbf{v}$ is the macroscopic velocity field.

$\hat{B}$ (here normalized so that $\hat{b}_0 = 1$) is usually expanded in Sonine polynomials,

$$\hat{B}(\mathcal{C}^2) = \sum_{n=0}^{\infty} \hat{b}_n S_{5/2}^{(n)}(\mathcal{C}^2). \tag{15}$$

If the gas also has a macroscopic velocity $\mathbf{v}$, we introduce its dimensionless counterpart as (in a monatomic gas the speed of sound is $\sqrt{5k_BT/3m}$)

$$\mathcal{V}_i = \sqrt{\frac{m}{2k_BT}}v_i = \sqrt{\frac{5}{6}}\text{Ma}. \tag{16}$$

In the distribution function (14), $\mathcal{C}_i$ is now replaced by $\mathcal{C}_i - \mathcal{V}_i$.

Let us first assume that the macroscopic velocity vanishes at the body. The influx of particles is then calculated from Eqs. (2) and (14). For symmetry reasons it can be written as

$$N_{(i)} = \left(1 + \frac{\beta_N}{2}\frac{p_{\langle ij\rangle}}{p}n_{\langle i}n_{j\rangle}\right)N_M, \tag{17}$$

$$\beta_N = \int_0^\infty \hat{B}(\mathcal{C}^2)\mathcal{C}^5 e^{-\mathcal{C}^2}d\mathcal{C} = \frac{1}{2}\int_0^\infty \hat{B}(x)x^2e^{-x}dx. \tag{18}$$

$\beta_N = 1$, if only the first term in the expansion (15) is retained. This is usually a good approximation.

For the influx of kinetic energy we similarly obtain

$$E_{(i)} = \left(1 + \frac{3\beta_E}{4}\frac{p_{\langle ij\rangle}}{p}n_{\langle i}n_{j\rangle}\right)E_M, \tag{19}$$

$$\beta_E = \frac{1}{6}\int_0^\infty \hat{B}(x)x^3e^{-x}dx. \tag{20}$$

β_E also takes the value 1 if just the lowest term is kept in the Sonine polynomials expansion.

V. EQUILIBRIUM TEMPERATURE IN SHEARING

A. Arbitrary body

We introduce a purely geometric tensor, characterizing the shape of the body,

$$\overline{n_{\langle i}n_{j\rangle}} = \frac{1}{S}\int (n_in_j - \tfrac{1}{3}\delta_{ij})ds.$$

The eigenvalues of the tensor $\overline{n_{\langle i}n_{j\rangle}}$ lie in the interval $[-1/3, 2/3]$. We note that $\overline{n_{\langle i}n_{j\rangle}}$ *vanishes for a sphere*.

From Eq. (11) we find the net total influx of kinetic energy as

$$\alpha_e\alpha_\tau\left[1 + \frac{3\beta_E}{4}\frac{p_{\langle ij\rangle}}{p}\overline{n_{\langle i}n_{j\rangle}} - \frac{T_w}{T}\left(1 + \frac{\beta_N}{2}\frac{p_{\langle ij\rangle}}{p}\overline{n_{\langle i}n_{j}}\right)\right]SE_M. \tag{21}$$

If the temperature of the body is kept the same as that of the gas, there is a nonvanishing net influx

$$\frac{\beta}{4}\alpha_e\alpha_\tau\frac{p_{\langle ij\rangle}}{p}\overline{n_{\langle i}n_{j\rangle}}SE_M. \tag{22}$$

Here ($\beta = 1$, if only the first term in the Sonine polynomial expansion is retained),

$$\beta = 3\beta_E - 2\beta_N. \tag{23}$$

As terms to second order in the mean free path are neglected we find the equilibrium temperature (13)

$$T_w = \left(1 + \frac{\beta}{4}\frac{p_{\langle ij\rangle}}{p}\overline{n_{\langle i}n_{j\rangle}}\right)T. \tag{24}$$

As $\overline{n_{\langle i}n_{j\rangle}}$ vanishes for a sphere, its equilibrium temperature is the same as that of the surrounding gas.

B. Axially symmetric bodies

For an axially symmetric body, with axis $\mathbf{N}$, the tensor $\overline{n_{\langle i}n_{j\rangle}} = aN_{\langle i}N_{j\rangle}$, where

$$a = \frac{3}{2}\frac{1}{S}\int\left[(\mathbf{n}\cdot\mathbf{N})^2 - \frac{1}{3}\right]ds. \tag{25}$$

For a flat body, $a = 1$. For a needle-shaped body, $a = -1/2$. For a sphere, $a = 0$.

Let us, in particular, consider an axially symmetric body in plane shear flow. In suitable coordinates, the only nonvanishing component of the velocity gradient is $v_{x,y} > 0$. Let us denote the polar angles (z is the polar axis) of the axial direction of the body by θ, φ. The equilibrium temperature is then

$$T_w = \left[1 + \frac{\beta a}{4}\frac{p_{xy}}{p}\sin^2\theta\sin(2\varphi)\right]T. \tag{26}$$

A needle-shaped body ($a = -1/2$) will have an equilibrium temperature, higher than that of the surrounding gas, if the axis of the body points into the first (or third) quadrant in the xy plane, but lower than that of the gas, if the axis points into the second (or fourth) quadrant.

If instead the body is kept at the temperature of the surrounding gas the total rate at which heat is transferred to the body from the gas is [E_M, the influx from a Maxwellian, is given by Eq. (7)]

$$\frac{\beta a}{4}\alpha_e\alpha_\tau\frac{p_{xy}}{p}\sin^2\vartheta\sin(2\varphi)E_MS. \tag{27}$$

Needle-shaped bodies absorb heat when their axes point into first (or third) quadrant. They emit heat when their axes point into the second (or fourth) quadrant. For flat bodies the situation is reversed.

VI. ADDING HOMOGENEOUS FLOW

When besides the shearing the gas also has a macroscopic velocity, we have, to the lowest order in the shearing, two contributions,

$$f \approx n\left(\frac{m}{2\pi k_BT}\right)^{3/2}\exp[-(\mathcal{C}_i - \mathcal{V}_i)^2] + n\left(\frac{m}{2\pi k_BT}\right)^{3/2}e^{-\mathcal{C}^2}\phi. \tag{28}$$

In the absence of shearing we find to second order in the Mach number,

$$\frac{\overline{N_{(i)}}}{N_M}=1+\mathcal{V}_n^2, \quad \frac{\overline{E_{(i)}}}{E_M}=(1+\mathcal{V}^2)+\frac{3}{2}\mathcal{V}^2\overline{n_{\langle i}n_{j\rangle}}e_ie_j\,.$$

Adding the two effects we have [**e** is the unit vector in the direction of the flow; Ma is given by Eq. (16)]

$$\frac{T_w}{T}=1+\frac{10}{9}\mathrm{Ma}^2+\left(\frac{5}{12}\mathrm{Ma}^2e_ie_j+\frac{\beta}{4}\frac{p_{\langle ij\rangle}}{p}\right)\overline{n_{\langle i}n_{j\rangle}}. \quad (29)$$

ACKNOWLEDGMENTS

It is a pleasure to acknowledge valuable discussions with Professor Y. Sone. This work was completed in Kyoto, a visit within the exchange program of the Swedish Academy of Sciences and the Japan Society for the Promotion of Science. It is a pleasure to thank Professor Aoki and Professor Sone for their kind hospitality. My thanks also go to K. Borg for reading the manuscript. This work was supported by the Swedish Research Council for Engineering Sciences.

[1] L.Z. Waldmann, Z. Naturforsch. A **14A**, 589 (1959).
[2] L.H. Söderholm, Rarefied Gas Dyn. **22**, 138 (2001).
[3] S. Bell and S.A. Schaaf, Jet Propul. **25**, 168 (1955).
[4] S. A. Schaaf, *Mechanics of Rarefied Gases* in *Handbuch der Physik* (Springer, Berlin, 1963), Vol. VIII/2, p. 591.
[5] G. A. Bird, *Molecular Gas Dynamics and the Direct Simulation of Gas Flows* (Oxford University Press, Oxford, 1994).
[6] C. Cercignani, *The Boltzmann Equation and Its Applications* (Springer-Verlag, New York, 1988).
[7] Y. Sone, J. Mec. Theor. Appl. **3**, 316 (1984).
[8] M. N. Kogan, *Rarefied Gas Dynamics* (Plenum, New York, 1969).
[9] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases*, 3rd ed. (Cambridge University Press, Cambridge, England, 1958).